# The approach to space charge limited current flow between coaxial cylinders



1975 J. Phys. A: Math. Gen. 8 555

(http://iopscience.iop.org/0305-4470/8/4/017)

J. Phys. A: Math. Gen., Vol. 8, No. 4, 1975. Printed in Great Britain. 

# The approach to space charge limited current flow between coaxial cylinders

C B Wheeler
Plasma Physics Department, Imperial College, London SW7 2AZ, UK

Received 24 October 1974, in final form 15 November 1974

**Abstract.** Poisson's equation in cylindrical symmetry is solved numerically in terms of the three following variables: the anode current, the cathode electric field and the ratio of the anode radius $r_1$ to the cathode radius $r_0$, where $r_1 > r_0$. The treatment is non-relativistic and is carried out for $1 \leqslant r_1/r_0 \leqslant 30\,000$ and for values of the cathode field ranging from 0–99% of the space charge free value. This analysis is particularly applicable to situations involving cathode field emission.

## 1. Introduction

Many years ago Langmuir (1913), Langmuir and Blodgett (1923, 1924) and Langmuir and Compton (1931) evaluated the fully space charge limited current flowing between coaxial cylinders and concentric spheres. These calculations, which assumed zero field at the cathode and an infinite supply of electrons of zero energy, have subsequently been applied to the design of thermionic values, electron guns and photoelectric cells. More recently Porter *et al* (1972) have investigated the effect of including the emission spectrum of the thermoelectrons in these geometries, and the earlier computations agree reasonably well with their results for full limitation provided that they are applied to the virtual cathode rather than to the physical cathode.

However there are many situations in plasma physics, particle physics and electron microscopy where the electron currents required greatly exceed those that can be produced by thermal or photoelectric emission. In such cases cathode field emission is employed, the field being enhanced by a cathode structure that is sharply curved. For this process the emission is very strongly dependent on the cathode field and there can be no virtual cathode since the field at the surface must always be large and positive. The electron emission is necessarily profoundly influenced by space charge effects and evaluation of the current requires simultaneous solution of the diode current/cathode field characteristics and the Fowler–Nordheim field emission relation for the relevant cathode material. This paper determines the diode current/cathode field characteristics in cylindrical geometry for cathode fields ranging from 0–99% of the space charge free value.

## 2. Mathematical formulation

Consider a coaxial cylindrical geometry of infinite length comprising an outer anode cylinder of radius $r_1$ maintained at a potential $V_1$ with respect to the inner cathode cylinder, radius $r_0$. In the steady state the potential and space charge density $\rho$ in the region $r_0 \leqslant r \leqslant r_1$ are related through Poisson's equation,

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}V}{\mathrm{d}r}\right) = -4\pi\rho. \tag{1}$$

If the electrons are emitted from the cathode with zero energy then $\rho$ can be related to the current $I$ per unit electrode length and also to the electron velocity $v$ which, in turn, can be related to the local potential:

$$I = -2\pi r\rho v, \qquad \tfrac{1}{2}mv^2 = eV. \tag{2}$$

This equation assumes that the potentials are sufficiently low to warrant a non-relativistic treatment and also that the current $I$ is sufficiently low for self-magnetic fields not to influence the electron motion. Equations (1) and (2) combine to give

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}V}{\mathrm{d}r}\right) = (2m/e)^{1/2}IV^{-1/2}. \tag{3}$$

It is convenient to express $I$ in terms of the current $I_0$ where

$$\frac{I_0}{2\pi r_1} = \frac{1}{9\pi}\left(\frac{2e}{m}\right)^{1/2} V_1^{3/2}r_1^{-2}. \tag{4}$$

Thus the right-hand side of this equation is the Child–Langmuir limiting current density that would flow between infinite parallel planes separated by a distance $r_1$ and at a potential difference $V_1$. Now redefine the position and potential variables in the following non-dimensional manner,

$$x = \ln(r/r_0), \qquad y = \left(\frac{9}{8}\frac{I_0}{I}\mathrm{e}^{x_1}\right)^{2/3}\frac{V}{V_1}, \tag{5}$$

where $x_1 = \ln(r_1/r_0)$. Equation (3) then reduces to

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \tfrac{1}{2}\mathrm{e}^x y^{-1/2}. \tag{6}$$

Multiplying throughout by $2\,\mathrm{d}y/\mathrm{d}x$ and integrating over $x \geqslant 0$ ($r \geqslant r_0$, $V \geqslant 0$, $y \geqslant 0$), then

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = A^2 + \int_0^y \mathrm{e}^x y^{-1/2}\,\mathrm{d}y \tag{7}$$

where $A = (\mathrm{d}y/\mathrm{d}x)_0$. Equation (5) shows that this parameter is proportional to the electric field $E$ at the cathode surface and it is instructive to relate this field to the field $E_0$ that would exist at the cathode surface in absence of space charge effects. Then

$$A = (y_1/x_1)(E/E_0) \qquad E_0 = V_1/r_0x_1. \tag{8}$$

$A = 0$ ($E = 0$) corresponds to fully space charge limited current flow and $A = \infty$ ($E = E_0$) corresponds to zero current and so to the absence of space charge effects. This latter limit follows from equation (5) which requires $y_1 = \infty$ at $I = 0$. Square-rooting equation (7), inverting and integrating over $0 \leqslant x \leqslant x_1$ ($r_0 \leqslant r \leqslant r_1$, $0 \leqslant V \leqslant V_1$, $0 \leqslant y \leqslant y_1$) gives

$$x_1 = \int_0^{y_1}\left(A^2 + \int_0^y e^x y^{-1/2}\,\mathrm{d}y\right)^{-1/2}\mathrm{d}y. \tag{9}$$

An analytic solution of this equation is only possible for small values of $x_1$, such that $e^{x_1} \simeq 1$, when

$$x_1 = (A^3/3)[(1+2y_1^{1/2}/A^2)^{3/2} - 3(1+2y_1^{1/2}/A^2)^{1/2} + 2] \qquad \text{for } x_1 < 1. \tag{10}$$

Substitution from equations (5) and (8) and rearrangement enable this equation to be expressed as a function of the two parameters $E/E_0$ and $x_1^2\,\mathrm{e}^{-x_1}I/I_0$:

$$\begin{aligned}(x_1^2\,\mathrm{e}^{-x_1}I/I_0)^{1/2} &= [1+\tfrac{9}{16}(E/E_0)^2(x_1^2\,\mathrm{e}^{-x_1}I/I_0)^{-1}]^{3/2} - \tfrac{27}{16}(E/E_0)^2(x_1^2\,\mathrm{e}^{-x_1}I/I_0)^{-1}\\ &\quad\times[1+\tfrac{9}{16}(E/E_0)^2(x_1^2\,\mathrm{e}^{-x_1}I/I_0)^{-1}]^{1/2}\\ &\quad+2[\tfrac{9}{16}(E/E_0)^2(x_1^2\,\mathrm{e}^{-x_1}I/I_0)^{-1}]^{3/2} \qquad \text{for } x_1 < 1.\end{aligned} \tag{11}$$

It is instructive to examine the meaning of these two parameters as $x_1 \to 0$. From equations (4), (5) and (8), with $r_1 - r_0 = d$, we have

$$\left.\begin{aligned} &x_1^{-2}\,\mathrm{e}^{x_1}I_0/2\pi r_1 \to (2e/m)^{1/2}V_1^{3/2}d^{-2}/9\pi \\ &E_0 \to V_1/d \end{aligned}\right\} \quad \text{as } x_1 \to 0.$$

Consequently these parameters respectively reduce to the ratio of the cathode field to the space charge free value and the ratio of the current to the Child–Langmuir limit, both for a plane geometry. Figure 1 shows a plot of equation (11) applied to the plane geometry; it is applicable to diodes of small curvature ($x_1 < 1$) simply by multiplying the ordinate by $x_1^2\,\mathrm{e}^{-x_1}$.

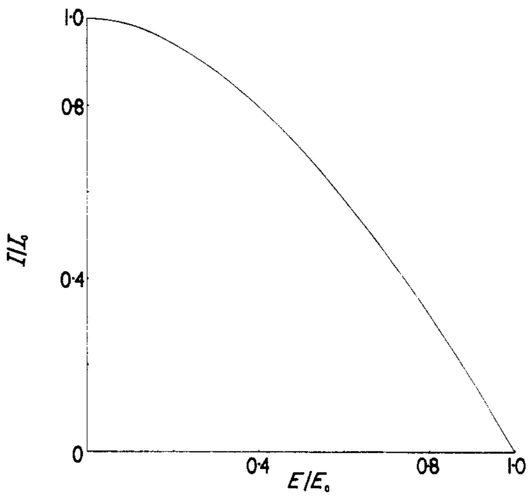


**Figure 1.** Dependence of anode current on cathode field for a plane diode.

## 3. Results and discussion

Equation (9) was integrated numerically by Simpson's rule employing an iterative technique that used equation (10) as a first approximation. The integration was performed for $0 \leqslant x_1 \leqslant 10{\cdot}3$ ($1 \leqslant r_1/r_0 \leqslant 30\,000$) and for 12 values of $A$ in the range $0 \leqslant A \leqslant 500$, yielding as many $(x_1, y_1)$ relations, each of which was transformed to a

$(r_1/r_0,\ I/I_0,\ E/E_0)$ relation using equations (5) and (8). This enabled 12 pairs of coordinates $(r_1/r_0,\ I/I_0)$ to be obtained for any specified value of $E/E_0$; figures 2 and 3 present the smooth curves through these coordinates. The general shape of all the

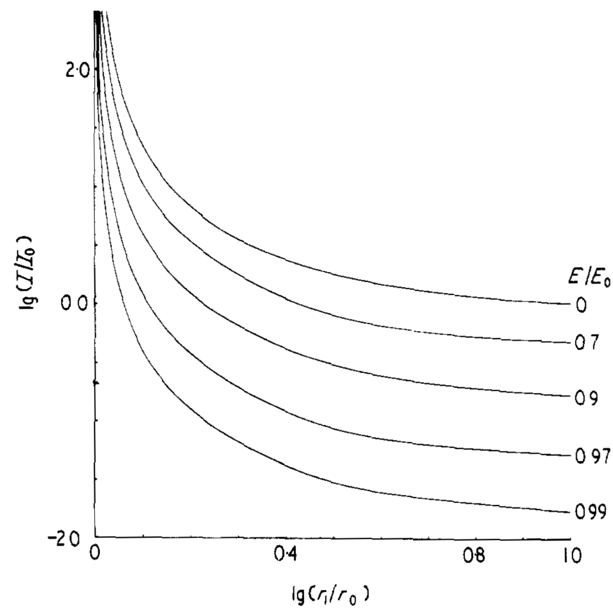


**Figure 2.** Dependence of anode current on electrode radii for selected values of cathode field.

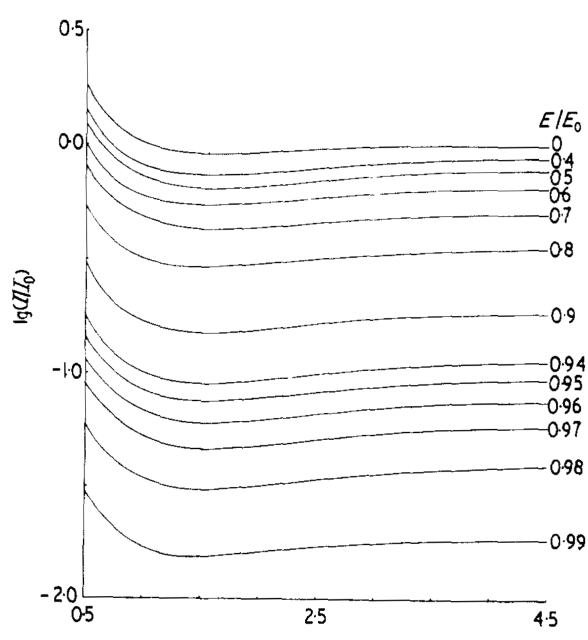


**Figure 3.** Dependence of anode current on electrode radii for selected values of cathode field.

curves is the same, namely an initial rapid fall of current from infinity followed by a current minimum in the vicinity of $20 < r_1/r_0 < 40$ and the approach to a near constant value as $r_1/r_0$ is increased further. For values of $r_1/r_0 < 1{\cdot}1$ in figure 2 the curves are represented analytically by equation (11), or by $I/I_0 = Bx_1^{-2}\,e^{x_1}$ where $B$ is the value of $I/I_0$ in figure 1 appropriate to the value of $E/E_0$ required. In the case of $E/E_0 = 0$ the values of $I/I_0$ correspond to the factor $\beta^{-2}$ tabulated by Langmuir and Blodgett (1923) and comparison shows agreement to better than 1% over the entire range of $r_1/r_0$ presented. A noticeable feature in figures 2 and 3 is the crowding together of the curves as $E/E_0$ decreases. For example the curves for $E/E_0 \leqslant 0{\cdot}3$ cannot be drawn on the scales presented since the values of $I/I_0$ deviate by as little as 10% from those corresponding to $E/E_0 = 0$. Comparison of figure 1 with figure 3 shows that, for large values of $r_1/r_0$, the values of $I/I_0$ do not differ by more than 10% for the plane and cylindrical cases. In this régime a useful criterion for the cathode field to be at least 99% of $E_0$ is that the current must not exceed 1% of $I_0$.

In applying these calculations to the evaluation of field emission currents a correction to the Fowler–Nordheim relation must be made due to the cathode curvature. This correction is negligible when the cathode field is in the range $10^6 < E < 10^8\ V\,\mathrm{cm}^{-1}$, as is the case for high current field emission devices. However, at very low fields the assumption that the cathode surface is a plane emitter can lead to calculated emission currents that are many orders of magnitude too great. The analysis of Sodha and Dubey (1969) shows that for a work function of 2·5 eV the plane assumption overestimates the emission current by a factor of $10^{12}$ at $E = 3\times10^4\ \mathrm{V\,cm^{-1}}$ and by $10^4$ at $E = 10^5\ \mathrm{V\,cm^{-1}}$.

## References

Langmuir I 1913 *Phys. Rev.* **2** 450–86
Langmuir I and Blodgett K B 1923 *Phys. Rev.* **22** 347–56
—— 1924 *Phys. Rev.* **24** 49–59
Langmuir I and Compton K T 1931 *Rev. Mod. Phys.* **3** 191–257
Porter J H, Franzen W and Wenstrup R S 1972 *J. Appl. Phys.* **43** 344–54
Sodha M S and Dubey P K 1969 *J. Phys.* D: *Appl. Phys.* **2** 1617–9